Scaling and Fluctuations of the Lyapunov Exponent in a 2D Anderson Localisation Problem

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(Received October 11, 2002)

KEYWORDS: localisation, Anderson model, fluctuation, scaling

In one-dimensional (1D) systems it is well known that the Lyapunov exponent (LE) has a normal distribution and that its average value \( \langle \gamma_{1D} \rangle \) is related to its variance \( \sigma_{1D}^2 \) by

\[ \frac{\sigma_{1D}^2 L}{\langle \gamma_{1D} \rangle} = 1. \] (1)

Here \( \langle \ldots \rangle \) represents a statistical average over realizations of the random potential. Expression (1) was first conjectured by Anderson et al.\textsuperscript{1} and later derived by many different authors within the framework of the random phase approximation. A correct and rigorous criterion for the validity of (1) was established only much later.\textsuperscript{2,3} For a sufficiently long 1D system the logarithm of the dimensionless conductance \( g \) is approximately

\[ \ln g \simeq -2\gamma L. \] (2)

Since a normal distribution is parameterised by its mean and variance, (1) establishes the single parameter scaling of the conductance distribution for 1D systems.

The objective of this paper is to establish a generalisation of (1) for the two dimensional Anderson model with diagonal disorder. We first investigate numerically the behaviour of the ratio on the l.h.s. of (1) in quasi-1D with diagonal disorder. We first investigate numerically

[Eq. 1]

\[ H = \sum_i \epsilon_i c_i^\dagger c_i - \sum_{i<j} c_i^\dagger c_j. \] (4)

Site energies \( \epsilon_i \) are uniformly distributed on the interval \([-W/2, W/2]\).

Before proceeding we must extend the usual definition of the LE, involving the taking of the limit \( L \to \infty \) to finite length \( L \). We consider a quasi-1D sample with the length \( L \) and width \( M \) \((L \gg M)\). In the transverse direction we impose periodic boundary conditions. Our definition takes as its starting point the transfer matrix method of MacKinnon and Kramer.\textsuperscript{4} We consider a transfer matrix \( T_L \) which is a product of a transfer matrix \( X_i \) for each slice up to the length \( L \), \( T_L = \prod_{i=1}^{L} X_i \). We prepare a random orthogonal \( 2M \times 2M \) matrix \( U_0 \). By repetition of a process involving several transfer matrix multiplication followed by a Gramm-Schmidt orthogonalization, we can express the matrix \( T_L U_0 \) as the product of an orthogonal matrix \( U_L \) and a right triangular matrix

\[ T_L U_0 = U_L \begin{pmatrix} D^{(1)}_L & R^{(1,2)}_L & \cdots & R^{(1,2M)}_L \\ 0 & D^{(2)}_L & \cdots & R^{(2,2M)}_L \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & D^{(2M)}_L \end{pmatrix} \] (5)

We define the Lyapunov exponents \( \gamma^{(i)}_L \) for finite length from the diagonal part of the right triangular matrix in (5)

\[ \gamma^{(i)}_L = \frac{1}{L} \ln D^{(i)}_L. \] (6)

In the present work, we concentrate on the statistics of the smallest positive Lyapunov exponent \( \gamma_L = \gamma^{(M)}_L \). The LE for finite length \( L \), defined in this way, is a random variable depending on the realisation of the random potential.

We study the dependence of the average \( \langle \gamma_L \rangle \) and its variance \( \sigma^2 \) on the strength of disorder \( W \), the energy \( E \), the width \( M \) and the length \( L \). The number of samples in each ensemble ranges from 1000 to 3000.

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We have taken \( L \) sufficiently large that the average \( \langle \gamma_L \rangle \) is independent of \( L \). In this limit its value is equal to the standard Lyapunov exponent defined in the limit \( L \to \infty \)
\[
\langle \gamma_L \rangle \simeq \gamma = \lim_{L \to \infty} \gamma_L. \quad (7)
\]
Thus the inverse of \( \langle \gamma_L \rangle \) is equal to the quasi-1D localisation length. Since, as is well known, this latter quantity obeys a one parameter scaling law we deduce that
\[
\langle \gamma_L \rangle M = F_\gamma \left( \frac{\xi}{M} \right) \quad (8)
\]
In the limit that \( M \gg \xi \) we expect that
\[
\langle \gamma_L \rangle M \to \frac{M}{\xi} \quad (9)
\]
Thus it seems reasonable to approximate the scaling function (8) by the expansion
\[
F_\gamma \left( \frac{\xi}{M} \right) = \frac{M}{\xi} + \sum_{n=0}^{n_\sigma} a_n \left( \frac{\xi}{M} \right)^n. \quad (10)
\]
Fitting our numerical data to this function, truncated at \( n_\sigma = 1 \), we obtain the localisation length \( \xi \) for each energy and disorder.

Next we consider the quantity \( \sigma^2 L \). For small \( L \), this depends on \( L \). However, we restrict attention here to \( L \) sufficiently large that \( \sigma^2 L \) becomes independent of \( L \) to within numerical accuracy. Since \( \langle \gamma_L \rangle \) is also independent of \( L \) in this limit, a one parameter scaling relationship of the form (3) between the mean and variance of LE is possible. Our numerical data are consistent with \( F_\sigma(\xi/M) \) approaching a constant value for large \( M/\xi \). Given this an expansion of the form
\[
F_\sigma \left( \frac{\xi}{M} \right) = \sum_{n=0}^{n_\sigma} b_n \left( \frac{\xi}{M} \right)^n \quad (11)
\]
is plausible. As none of the expansion coefficients is fixed, the absolute value of \( \xi \) and the fitting parameters \( b_n \) cannot be determined by fitting only to (11). To obtain their absolute values, we fix the localisation length at \( E = 0.0 \) and \( W = 7.0 \) as \( \xi = 20.63 \) according to the result of finite size scaling analysis of \( \langle \gamma_L \rangle M \) presented above. After that we use (11), truncated at \( n_\sigma = 4 \), to find the absolute values of the coefficients \( b_n \) and the two-dimensional localisation length \( \xi \) for each value of energy and disorder.

The data and scaling function \( F_\sigma(\xi/M) \) are shown in Fig. 1. It is seen that, within the accuracy of the simulation, all the data fall on a single curve confirming our assumption of a one parameter scaling for the variance as described by (3). The estimates of \( \xi \) obtained from the two analyses, based on the scaling of \( \sigma^2 L/\langle \gamma_L \rangle \) and of \( \langle \gamma_L \rangle M \), are in close agreement. This finding is strong evidence that the distribution of the LE in the two-dimensional Anderson model is described by a single parameter.

With decreasing \( M/\xi \), \( \sigma^2 L/\langle \gamma_L \rangle \) appears to approach unity consistent with the relation (1) for 1D. For large

\[
M/\xi, \sigma^2 L/\langle \gamma_L \rangle \text{ approaches the asymptotic value } b_0. \text{ We estimate}
\]
\[
b_0 = \lim_{M/\xi \to \infty} F_\sigma \left( \frac{\xi}{M} \right) = 0.13 \pm 0.01. \quad (12)
\]
This value is significantly smaller than the value of unity for one-dimensional systems indicating that the fluctuations in 2D systems are much weaker than in 1D systems.